The Poisson equation on Riemannian manifolds with weighted Poincaré inequality at infinity

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Nonlinear Geometric PDE's
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joint work with G. Catino and F. Punzo (PoliMi)

The Poisson equation

Let (M, g) be a complete Riemannian manifold with empty boundary, $\partial M = \emptyset$, $n = \dim(M)$. Let $f : M \to \mathbb{R}$ be a (regular) function. Then u is a (classical) solution to the *Poisson equation* if

$$\Delta u = f \qquad \text{on } M,$$

where Δ denotes the Laplace-Beltrami operator, i.e. $\Delta = \text{trace}\left(\nabla^2\right) = \text{div}(\nabla)$.

- If M is compact, then there exists a solution if and only if $\int_M f = 0$.
- If (M, g) is rotationally symmetric (e.g. $\mathbb{R}^n$ or $\mathbb{H}^n$) sufficient (and necessary) conditions for the existence of (radial) solutions can be found by solving an associated ODE.

Question: find natural conditions on the manifold (M, g) and the function f to guarantee the *existence* of a solution u to the Poisson equation.

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Question: find natural conditions on the manifold (M, g) and the function f to guarantee the *existence* of a solution u to the Poisson equation.

Green's function

As it is well-known, the solvability of the Poisson equation is closely related to the existence of the so-called *Green's function*.

Malgrange ('55) showed that a complete Riemannian manifold (M, g) always admits a Green's function $G(x, y)$, namely a symmetric function satisfying

$$\Delta G(x, \cdot) = -\delta_x(\cdot) \qquad \text{on } M.$$

In particular, if $f \in C_0^\infty(M)$, then a solution u to the Poisson equation exists and is given by

$$u(x) = \int_M G(x, y) f(y)\, dy\,.$$

A good control on a Green's function will enable to establish existence of solutions u for more general functions f.

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Notation and preliminaries

Let (M, g) be a complete non compact Riemannian manifold without boundary, $n = \dim(M)$.

Fix a reference point $p \in M$ and denote by $r(x) = \mathrm{dist}(x, p)$. For any $x \in M$ and $R > 0$, we denote by $B_R(x)$ the geodesic ball of radius R with center x.

We denote by Ric the Ricci curvature of g.

- By definition (M, g) is *non-parabolic* if it admits a *positive* Green's function, and *parabolic* otherwise. If the manifold is non-parabolic, there exists a unique *minimal* positive Green's function. Equivalently, (M, g) is non-parabolic if it admits a non-constant positive superharmonic function.
- Let $\lambda_1(M)$ be the bottom of the L^2-spectrum of the Laplace operator $-\Delta$. One has $\lambda_1(M) \geq 0$. Moreover if $\lambda_1(M) > 0$ then (M, g) is non-parabolic.

(A very good reference is [Grigor'yan, '99])

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Poisson equation: previous results

There exists a solution u to the Poisson equation $\Delta u = f \in C^{\alpha}_{loc}(M)$ if

- [Strichartz, '83]: $\lambda_1(M) > 0$ and $f \in L^p(M)$, for some $1 < p < \infty$.
- [Ni, '02]: $\lambda_1(M) > 0$ (non-parabolic) and $f \in L^1(M)$.
- [Ni-Shi-Tam, '01]: $Ric \geq 0$ and

$$|f(x)| \leq \frac{C}{\big(1 + r(x)\big)^{2+\varepsilon}}$$

  for some $C > 0$ and $\varepsilon > 0$. An integral assumption involving averages of f is sufficient. Sharp on $\mathbb{R}^n$.

- [Munteanu-Sesum, '10]: $\lambda_1(M) > 0$, $\mathrm{Ric} \geq -K$ and

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A first result

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Then for any $V \in T_xM$ with $|V| = 1$, one has $\text{Ric}(V, V)(x) \geq \mu(x)$, and we have $\mu(x) \geq -\omega(r(x))$ for some $\omega \in C([0, \infty))$, $\omega \geq 0$. Hence, for any $x \in M$, by solving a simple ODE, we have

$$\text{Ric}(V, V)(x) \geq -(n-1)\frac{\varphi''(r(x))}{\varphi(r(x))},$$

for some $\varphi \in C^\infty((0, \infty)) \cap C^1([0, \infty))$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$. Note that $\varphi, \varphi', \varphi''$ are positive in $(0, \infty)$. For a fixed small $\varepsilon_0 > 0$ (depending on the geometry of the manifold) we set

$$\tilde{K}(R) := \sup_{y \in B_R(p) \setminus B_{\varepsilon_0}(p)} \frac{\varphi''(r(y))}{\varphi(r(y))}, \quad \hat{K}(R) := \sup_{y \in B_R(p) \setminus B_{\varepsilon_0}(p)} \frac{\varphi'(r(y))}{\varphi(r(y))},$$

$$K(R) := \max\{1, \tilde{K}(R), \hat{K}(R)\}.$$

Then we define

$$\theta(R) := R\sqrt{K(R)}.$$

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- G. Catino, D.D. M., F. Punzo, *The Poisson equation on manifolds with positive essential spectrum*, 2018, preprint.

Theorem [Catino-M.-Punzo, '18]

Let (M, g) be a complete noncompact Riemannian manifold with $\lambda_1(M) > 0$. Suppose that f is a locally Hölder function on M. If

$$\sum_{j=1}^{\infty} \frac{\theta(j+1) - \theta(j)}{\lambda_1\left(M \setminus B_j(p)\right)} \cdot \sup_{M \setminus B_j(p)} |f| < \infty,$$

then $\Delta u = f$ has a classical solution u.

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Remarks about the assumptions

$$\lambda_1(M) > 0 \qquad \text{and} \qquad \sum_{j=1}^{\infty} \frac{\theta(j+1) - \theta(j)}{\lambda_1(M \setminus B_j(p))} \cdot \sup_{M \setminus B_j(p)} |f| < \infty .$$

- With some (nontrivial) work we can replace the assumption $\lambda_1(M) > 0$ with a weaker one, namely we can assume positivity of the *essential spectrum*, $\lambda_1^{\mathrm{ess}}(M) > 0$, which is equivalent to say that $\lambda_1(M \setminus K) > 0$, for some compact subset $K \subset M$.
- $\theta(j+1) - \theta(j)$ is related to a lower bound of the Ricci curvature (and to an upper bound for $\Delta r(x)$).
- In particular, if $\mathrm{Ric} \geq -K$, then $\theta(j+1) - \theta(j) \leq C$ for any j.
  By monotonicity $\lambda_1(M \setminus B_j(p)) \geq \lambda_1(M)$.
  Hence, we recover Munteanu-Sesum's result.
- The result is *sharp*, on a family of *rotationally symmetric (model) manifolds*.
- Main drawbacks of this result: the spectral assumption $\lambda_1^{\mathrm{ess}}(M) > 0$ places some strong conditions on the geometry of the ambient manifold, and the geometry of all the manifold is relevant (while one would like to have conditions only on the geometry of the manifold *"at infinity"*, i.e. outside an arbitrarily large compact set).

Remarks about the assumptions

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Corollaries

Corollary

Let (M, g) be a complete noncompact Riemannian manifold with $\lambda_1^{\text{ess}}(M) > 0$ and let f be a locally Hölder function on M. If

$$\text{Ric} \geq -C\big(1 + r(x)\big)^{\gamma}, \qquad |f(x)| \leq \frac{C}{\big(1 + r(x)\big)^{1+\frac{\gamma}{2}+\varepsilon}},$$

for some $C > 0$, $\gamma \geq 0$ and $\varepsilon > 0$, then $\Delta u = f$ has a classical solution u.

Corollary

Let (M, g) be a Cartan-Hadamard manifold and let f be a locally Hölder function on M. If

$$-\frac{1}{C}\big(1 + r(x)\big)^{\gamma_1} \leq \text{Ric}(x) \leq -C\big(1 + r(x)\big)^{\gamma_2}, \quad |f(x)| \leq \frac{C}{\big(1 + r(x)\big)^{1+\frac{\gamma_1}{2}-\gamma_2+\varepsilon}},$$

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The Poisson eq. with a weighted Poincaré ineq.

We say that (M, g) satisfies a *weighted Poincaré inequality* with a positive weight function ρ if

$$\int_M \rho(x)\, v^2(x)\, dV \leq \int_M |\nabla v|^2\, dV$$

for every $v \in C_c^\infty(M)$. Moreover, (M, g) satisfies the property $(\mathcal{P}_\rho)$ if a weighted Poincaré inequality holds for the weight ρ and the conformal ρ-metric $g_\rho := \rho\, g$ is complete. Note that the *"best constant"* in the inequality is normalized to 1.

Some examples:

- (M, g) with positive spectrum: $\rho := \lambda_1(M) > 0$. $(\mathcal{P}_\rho)$ holds.
- $(\mathbb{R}^n, g_{\text{flat}})$: $\rho(x) := \left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2}$ *(Hardy inequality)*. $(\mathcal{P}_\rho)$ holds.
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- G. Catino, D.D. M., F. Punzo, *The Poisson equation on Riemannian manifolds with a weighted Poincaré inequality at infinity*.

Theorem 1 [Catino-M.-Punzo]

Let (M, g) be a complete non-compact non-parabolic Riemannian manifold with minimal positive Green's function G. Let $\rho(x) := \frac{|\nabla G(p,x)|^2}{4G^2(p,x)}$ and let f be a locally Hölder continuous function on M. If

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then the Poisson equation $\Delta u = f$ admits a classical solution u.

- $\omega(j)$ is a refinement of the function $\theta(j)$ (and it is *increasing*). On $\mathbb{R}^n$

$$\omega(j+1) - \omega(j) = C \log\left(1 + \frac{1}{j}\right) \sim \frac{C}{j}$$

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Theorem 1 [Catino-M.-Punzo]

Let (M, g) be a complete non-compact non-parabolic Riemannian manifold with minimal positive Green's function G. Let $\rho(x) := \frac{|\nabla G(p,x)|^2}{4G^2(p,x)}$ and let f be a locally Hölder continuous function on M. If

$$\sum_{j=1}^{\infty} (\omega(j+1) - \omega(j)) \sup_{M \setminus B_j(p)} \left| \frac{f}{\rho} \right| < \infty,$$

then the Poisson equation $\Delta u = f$ admits a classical solution u.

- $\omega(j)$ is a refinement of the function $\theta(j)$ (and it is *increasing*). On $\mathbb{R}^n$

$$\omega(j+1) - \omega(j) = C \log\left(1 + \frac{1}{j}\right) \sim \frac{C}{j}$$

- The result is sharp on $\mathbb{H}^n$, on $\mathbb{R}^n$ and on (a family of) model manifolds.

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What happens with a *weighted Poincaré inequality* for functions with compact support, outside a fixed compact set?

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Technical remarks

With the same notation as before, we define for $r(x) > R > 1$

$$K_R(x) := \sup_{y \in B_{r(x)+R}(p) \setminus B_{r(x)-R}(p)} \frac{\varphi''(r(y))}{\varphi(r(y))},$$

$$I_R(x) := \begin{cases} \sqrt{K_R(x)} \coth\left(\sqrt{K_R(x)} R/2\right) & \text{if } K_R(x) > 0, \\ \frac{2}{R} & \text{if } K_R(x) = 0 \end{cases}$$

$$Q_R(x) := \max\left\{K_R(x), \frac{I_R(x)}{R}, \frac{1}{R^2}\right\},$$

$$\omega(x) = \omega(r(x)) := \int_1^{r(x)} \sqrt{Q_{r(\gamma(s))/4}(r(\gamma(s)))}\, ds.$$

where γ is a minimal geodesic connecting p to x.

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The result holds under the more general assumption: for every $m \in \mathbb{N}$ sufficiently large there exists a positive weight function ρ_m such that

$$\int_{M \setminus B_m(p)} \rho_m(x)\, v^2(x)\, dV \leq \int_{M \setminus B_m(p)} |\nabla v|^2\, dV$$

for every $v \in C_c^\infty(M \setminus B_m(p))$. Moreover, the conformal ρ_m-metrics

$$g_{\rho_m} := \rho_m\, g$$

are complete.

A key property: if $r_{\rho_m}(x) := dist_{\rho_m}(x, p)$ then

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Notice that, in the more general case where one uses a family of weight functions ρ_m, the manifold can be parabolic and we have to treat this case separately.

In order to prove the theorem, we have to show that, for every $x \in M$,

$$\left| \int_M G(x,y) f(y)\, dy \right| < \infty.$$

This will define the function $u(x)$ solution to the Poisson equation $\Delta u = f$.

Using the *Harnack inequality* one sees that the above integral is finite at any point $x \in M$ *if and only if* it is finite at p.

For any $0 \leq a < b \leq \infty$, we define

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Preliminary estimates I

The Green's function $G(p,\cdot)$ is harmonic away from p. In particular we can apply the following

Lemma 1: Local gradient estimate for harmonic function [Yau]

Let $R > 0$ and $z \in M$ with $r(z) > R$. Let $w \in C^2$ be positive and harmonic away from p. Then

$$|\nabla w(z)| \leq C\sqrt{Q_R(z)}\, w(z).$$

As a consequence, integrating along geodesics and using Gronwall inequality one obtains

Lemma 2: Local pointwise estimate for $G(x,y)$

Let $y \in M$ with $r(y) > a \geq 1$. Then

$$A^{-1}e^{-B\omega(y)} \leq G(p,y) \leq A\, e^{B\omega(y)}.$$

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Lemma 2 (pointwise estimate) and the maximum principle imply

$$\mathcal{L}_p\left(0, A^{-1}e^{-B\omega(a)}\right) \subset M \setminus B_a(p),$$

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Using heat kernel estimates [Li-Yau] and volume comparison we can show

Prop. 1: High-level sets

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Under the hypotheses of Theorem 1 we have the following

Prop. 2: Intermediate-level sets (for Theorem 1)

There exists a positive constant C such that, for any locally Holder continuous function f, $0 < \delta < 1$ and $\varepsilon > 0$,

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Basically a consequence of the coarea formula.

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Under the hypotheses of Theorem 2, we have the weaker result

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Remarks:

- This estimate was essentially proved in [Li-Wang, '06] and used in [Munteanu-Sesum, '10].
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$$\left|\int_M G(p,y)f(y)\,dy\right| \le \int_{\mathcal{L}_p\left(0,\,A\,e^{B\omega(a)}\right)} G(p,y)|f(y)|\,dy + \int_{\mathcal{L}_p\left(A\,e^{B\omega(a)},\infty\right)} G(p,y)|f(y)|\,dy\,.$$

Since f is locally bounded, the second term is controlled by Prop. 1 (high-level sets). To estimate the first term, suitably choose a sequence $a_m \searrow 0^+$ starting at $a_0 = A\,e^{B\omega(a)}$ and note that

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From Prop. 2 (intermediate-level sets):

$$\int_{\mathcal{L}_p(a_{m+1},a_m)} G(x,y)|f(y)|\,dy \le C\left(\log\frac{a_m}{a_{m+1}}\right)\sup_{\mathcal{L}_p(a_{m+1},a_m)}\left|\frac{f}{\rho}\right|.$$

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Hence we obtain (for large m)

$$\int_{\mathcal{L}_p(a_{m+1},a_m)} G(x,y)|f(y)|\,dy \le C(\omega(m+1)-\omega(m)) \sup_{M\setminus B_m(p)} \left|\frac{f}{\rho}\right| .$$

Thus,

$$\int_{\mathcal{L}_p\left(0,\,A\,e^{B\omega(a)}\right)} G(x,y)|f(y)|\,dy$$
$$\le C_1 \sum_{m=1}^{\infty} (\omega(m+1)-\omega(m)) \sup_{M\setminus B_m(p)} \left|\frac{f}{\rho}\right| + C_2 < +\infty.$$

Then we finally conclude

$$\left|\int_M G(p,y) f(y)\,dy\right| < \infty$$

and $u(x)$ is a (classical) solution.

Sketch of the proof IV

Hence we obtain (for large m)

$$\int_{\mathcal{L}_p(a_{m+1},a_m)} G(x,y)|f(y)|\,dy \leq C(\omega(m+1)-\omega(m)) \sup_{M\setminus B_m(p)} \left|\frac{f}{\rho}\right| .$$

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Thank you!

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Sharpness: model manifolds I

Let (M, g) be a rotationally symmetric manifold with pole $p \in M$ and metric given by

$$g = dr^2 + \varphi(r)^2 g_{\mathbb{S}^{n-1}}$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on the round sphere $\mathbb{S}^{n-1}$ and $\varphi \in C^\infty((0,\infty)) \cap C^1([0,\infty))$ with $\varphi > 0$ if $r > 0$, $\varphi(0) = 0$ and $\varphi'(0) = 1$.

The choice $\varphi(r) = r$ and $\varphi(r) = \sinh(r)$ gives the standard metrics on $\mathbb{R}^n$ and $\mathbb{H}^n$, respectively. The Laplacian is given by

$$\Delta = \partial_r^2 + (n-1)\frac{\varphi'}{\varphi}\partial_r + \frac{1}{\varphi^2}\Delta_{\mathbb{S}^{n-1}}.$$

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Sharpness: model manifolds II

Let $\gamma \in \mathbb{R}$ and choose for $r \gg 1$

$$\varphi(r) = \begin{cases} \exp\left(B\, r^{1+\frac{\gamma}{2}}\right) & \text{if } \gamma > -2 \\ r^{\delta} & \text{if } \gamma = -2 \\ r & \text{if } \gamma < -2 \end{cases}$$

extended suitably near $r = 0$ (B, δ suitable positive constants). Then

$$\mathrm{Ric} \geq -C(1+r)^{\gamma}.$$

Choose $f = f(r) = 1/(1+r)^{\alpha}$. With this choices, it is easy to see that the integral defining $u(p)$ is finite (so a solution exists) if and only if

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